Angular-momentum nonclassicality by breaking classical bounds on statistics

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We derive simple practical procedures revealing the quantum behavior of angular momentum variables by the violation of classical upper bounds on the statistics. Data analysis is minimum and definite conclusions are obtained without evaluation of moments, or any other more sophisticated procedures. These nonclassical tests are very general and independent of other typical quantum signatures of nonclassical behavior such as sub-Poissonian statistics, squeezing, or oscillatory statistics, being insensible to the nonclassical behavior displayed by other variables.

I. INTRODUCTION

Nonclassicality is a key concept supporting the necessity of the quantum theory [1–8]. A customary signature of nonclassical behavior is the failure of the Glauber-Sudarshan P phase-space representation to exhibit all the properties of a classical probability density. This occurs when P takes negative values, or when it is more singular than a delta function.

In a recent work we have derived exceedingly simple and robust practical procedures to reveal the quantum nature of states and measurements [9, 10]. These are upper bounds on the outcome probabilities which are satisfied when the P representative is compatible with classical physics. The lack of compliance of these statistical bounds is thus a nonclassical signature so this provides sufficient, not necessary, criteria of nonclassicality.

In this work we derive the classical upper bounds for the statistics of angular momentum or spin components, this is to say SU(2) variables. They are derived in terms of the classical or nonclassical behavior of the SU(2) P function for states and measurements. This generalizes previous particular examples considered in Ref. [9]. For definiteness we focus on quantum optics where SU(2) variables represent very basic items such as polarization and two-beam interference. The main properties of this approach are:

i) The violation of these bounds can be ascribed exclusively to the nonclassical behavior of SU(2) variables, this is when the SU(2) P function takes negative values or is more singular than a delta function, irrespective of the classical or nonclassical behavior of other variables, such as light intensity (photon number).

ii) We show that these SU(2) upper bounds are larger than the ones derived from the quadrature P function. In the bright limit they coincide with the bounds for field quadratures.

iii) The only previously reported nonclassical spin property is SU(2) squeezing [11–14] (in passing we explicitly demonstrate below that SU(2) squeezing is actually an SU(2) nonclassical property). This approach generalizes and simplifies the idea of SU(2) squeezing so that it can be easily applied to any spin observable. This is achieved without involving state reconstruction, i. e., without complete knowledge of the SU(2) P function or any other distribution [15, 16].

iv) Data analysis is reduced to minimum so that definite conclusions can be obtained without evaluation of moments, or any other more sophisticated data elaborations [1–6]. This is reflected on the robustness under practical imperfections [9, 10].

v) These nonclassical tests are in general independent of other typical quantum signatures such as sub-Poissonian statistics, squeezing, or oscillatory statistics [1]. To show this we provide some examples of quantum states violating classical bounds that present no such typical nonclassical signatures.

In Sec. II we recall the main tools required to the quantum description of angular-momentum variables, including SU(2) squeezing and the classical upper bounds to the statistics of arbitrary spin observables. In Sec. III we show that the angular-momentum components are nonclassical observables. We also derive the classical upper bounds for the statistics of angular-momentum components, applying them to some relevant states.

II. SU(2) SYSTEMS

In this section we first recall basic material on SU(2) states and observables relevant for the analysis of their nonclassical properties. We also demonstrate that SU(2) squeezing is actually an SU(2) nonclassical property.

A. Angular momentum operators

Arbitrary dimensionless angular momentum operators $\boldsymbol{j} = (j_1, j_2, j_3)$ satisfy the commutation relations

$$[j_k, j_\ell] = i \sum_{n=1}^{3} \epsilon_{k,\ell,n} j_n, \qquad [j_0, \boldsymbol{j}] = \boldsymbol{0}, \tag{2.1}$$

where $\epsilon_{k,\ell,n}$ is the fully antisymmetric tensor with $\epsilon_{1,2,3} = 1$, and j_0 is defined by the relation

$$\boldsymbol{j}^2 = j_0 (j_0 + 1). \tag{2.2}$$

Note that this implies that all quantities to be considered throughout this work, including all plots, are dimensionless.

For the sake of completeness we take into account that j_0 may be an operator. This is the case of two-mode bosonic realizations where j_0 is proportional to the number of particles. More specifically

$$j_0 = \frac{1}{2}\left(a_1^\dagger a_1 + a_2^\dagger a_2\right), \quad j_1 = \frac{1}{2}\left(a_2^\dagger a_1 + a_1^\dagger a_2\right),$$
$$j_2 = \frac{i}{2}\left(a_2^\dagger a_1 - a_1^\dagger a_2\right), \quad j_3 = \frac{1}{2}\left(a_1^\dagger a_1 - a_2^\dagger a_2\right), \tag{2.3}$$

where $a_{1,2}$ are the annihilation operators of two independent bosonic modes with $[a_j, a_j^\dagger] = 1$, $[a_1, a_2] = [a_1, a_2^\dagger] = 0$ [17]. We have the following correspondence

$$|j, m\rangle = |n_1 = j + m\rangle |n_2 = j - m\rangle, \tag{2.4}$$

between the $|j, m\rangle$ basis of simultaneous eigenvectors of j_3 and j_0, with $j_3|j, m\rangle = m|j, m\rangle$ and $j_0|j, m\rangle = j|j, m\rangle$, and the product of two-mode number states $|n_1\rangle|n_2\rangle$, with $a_j^\dagger a_j|n_j\rangle = n_j|n_j\rangle$. *The quantum number j represents the total number of bosons. For most realistic and practical situations the number of bosons usually rather large, so below we will consider suitable approximations of results in the limit $j \gg 1$.*

Concerning physical realizations, $a_{1,2}$ can represent the complex amplitude operators of two electromagnetic field modes. The operators $\boldsymbol{j}$ describe the polarization of transverse electromagnetic waves (representing the Stokes operators) as well as two-beam interference. For material systems $a_{1,2}$ can represent the annihilation operators for two species of atoms in two different internal states, for example. Angular momentum operators also serve to describe the internal state of two-level atoms via the definitions

$$j_0 = \frac{1}{2}\left(|e\rangle\langle e| + |g\rangle\langle g|\right), \quad j_1 = \frac{1}{2}\left(|g\rangle\langle e| + |e\rangle\langle g|\right),$$
$$j_2 = \frac{i}{2}\left(|g\rangle\langle e| - |e\rangle\langle g|\right), \quad j_3 = \frac{1}{2}\left(|e\rangle\langle e| - |g\rangle\langle g|\right), \tag{2.5}$$

where $|e, g\rangle$ are the excited and ground states. This is formally an spin 1/2 where $j_{0,3}$ represent atomic populations and $j_{1,2}$ the atomic dipole [18]. Collections of two-level atoms are described by composition of the individual angular momenta. We recall that for spin 1/2 spin nonclassicality is equivalent to entanglement [19].

B. Phase space representatives

The SU(2) Q and P functions associated to an operator A are defined after the SU(2) coherent states $|j, \Omega\rangle$ [20]

$$|j, \Omega\rangle = \sum_{m=-j}^{j} \begin{pmatrix} 2j \\ m+j \end{pmatrix}^{1/2} \sin^{j-m}\left(\frac{\theta}{2}\right) \cos^{j+m}\left(\frac{\theta}{2}\right) \exp[-i(j+m)\phi] |j, m\rangle, \tag{2.6}$$

with $\pi \geq \theta \geq 0$, and $\pi \geq \phi \geq -\pi$, as

$$A = \int d^2\Omega P(\Omega) |j, \Omega\rangle\langle j, \Omega|, \quad Q(\Omega) = \frac{2j+1}{4\pi}\langle j, \Omega|A|j, \Omega\rangle, \tag{2.7}$$

with $d^2\Omega = \sin\theta d\theta d\phi$. They are suitably normalized since

$$\int d^2\Omega P(\Omega) = \int d^2\Omega Q(\Omega) = \mathrm{tr} A. \tag{2.8}$$

Arbitrary measurements are described by positive operator-valued measures (POVMs) Δ_k, such that the probability of the outcome k is $p_k = \mathrm{tr}(\Delta_k \rho)$, where ρ is the measured state. In terms of the SU(2) phase-space representatives the statistics can be expressed as

$$p_k = \frac{4\pi}{2j+1}\int d^2\Omega P_k(\Omega) Q(\Omega) = \frac{4\pi}{2j+1}\int d^2\Omega P(\Omega) Q_k(\Omega), \tag{2.9}$$

where $P(\Omega)$ and $Q(\Omega)$ are the representatives of the measured state ρ, while $P_k(\Omega)$ and $Q_k(\Omega)$ are the ones associated to the POVM Δ_k.

We say that the measurement is nonclassical when the P representative of some Δ_k takes negative values or is more singular than a delta function. In most practical situations Δ_k define legitimate measuring states $\rho_k \propto \Delta_k$ so that the measurement is nonclassical if and only there is a nonclassical measuring state ρ_k.

C. SU(2) squeezing

This can be regarded as the first exclusively SU(2) nonclassical property. In general terms, the idea of SU(2) squeezing means reduced fluctuations below the level established by the SU(2) coherent states [20]. There are several quantitative implementations of this idea [11–14]:

i) The less stringent squeezing criterion is that the fluc-

tuations of a $\boldsymbol{j}$ component $j_\perp$ orthogonal to the direction of $\langle \boldsymbol{j} \rangle$ (this is that $\langle j_\perp \rangle = 0$) must be lesser than in a SU(2) coherent state, leading to

$$(\Delta j_\perp)^2 < \frac{j}{2}. \qquad (2.10)$$

ii) SU(2) squeezing can be defined as equivalent to provide larger interferometric resolution than coherent states, leading to

$$\frac{(\Delta j_\perp)^2}{\langle \boldsymbol{j} \rangle^2} < \frac{1}{2j}. \qquad (2.11)$$

This implies the satisfaction of the most general squeezing condition (2.10).

iii) Finally, there is also the idea of squeezing derived from the uncertainty relations (focusing again on orthogonal components)

$$\Delta j_{\perp,1} \Delta j_{\perp,2} \geq \frac{1}{2} |\langle \boldsymbol{j} \rangle|, \qquad (2.12)$$

so that SU(2) squeezing would mean

$$(\Delta j_\perp)^2 < \frac{|\langle \boldsymbol{j} \rangle|}{2}, \qquad (2.13)$$

which implies the satisfaction of both Eqs. (2.10) and (2.11). In particular, this is achieved by the SU(2) intelligent states determined by the following eigenvalue equation [13]

$$(\eta j_{\perp,2} + i j_{\perp,1}) |\psi\rangle = 0, \qquad (2.14)$$

where η is a real parameter. For $\eta = 1$ they are SU(2) coherent states so that uncertainty-relations squeezing (2.13) occurs for $\eta \neq 1$ and implies the satisfaction of the other criteria (2.10) and (2.11).

1. *SU(2) squeezing is an SU(2) nonclassical property*

Next we show that every SU(2) squeezed state has a nonclassical SU(2) P distribution. This completes the proof in Ref. [14] where it was shown in bosonic realizations that SU(2) squeezing implies nonclassical quadrature P function.

To this end we focus on the most general criterion in Eq. (2.10). Using the SU(2) P representation we have

$$(\Delta j_\perp)^2 = \langle j_\perp^2 \rangle = \int d^2\Omega P(\Omega) \langle j, \Omega | j_\perp^2 | j, \Omega \rangle. \qquad (2.15)$$

It can be easily seen using SU(2) invariance that for any component $j_u = \boldsymbol{u} \cdot \boldsymbol{j}$ with $\boldsymbol{u}^2 = 1$ we have the identity

$$\langle j, \Omega | j_u^2 | j, \Omega \rangle = \frac{j}{2} + \frac{2j-1}{2j} \langle j, \Omega | j_u | j, \Omega \rangle^2. \qquad (2.16)$$

To demonstrate this relation we use SU(2) invariance (every SU(2) coherent state can be obtained by applying an SU(2) transformation to $|j, m = j\rangle$) so that

$$\langle j, \Omega | j_u^k | j, \Omega \rangle = \langle j, m = j | j_v^k | j, m = j \rangle \qquad (2.17)$$

where $|j, m = j\rangle$ is in the j_0, j_3 basis and $\boldsymbol{v}$ is a unit real vector related with $\boldsymbol{u}$ by a rotation. Using the bosonic representation (2.3) the state $|j, m = j\rangle$ becomes the photon number state $|n\rangle|0\rangle$ so that $\langle j, m = j | j_v | j, m = j \rangle = v_3 n/2$ and

$$\begin{aligned}\langle j, m = j | j_v^2 | j, m = j \rangle &= (v_1^2 + v_2^2)\frac{n}{4} + v_3^2 \frac{n^2}{4} \qquad (2.18)\\ &= \frac{n}{4} + v_3^2 \frac{n^2}{4}\left(1 - \frac{1}{n}\right),\end{aligned}$$

where $v_{1,2,3}$ are the components of $\boldsymbol{v}$. This leads to Eq. (2.16) after some simple algebra.

Therefore, for arbitrary states

$$(\Delta j_\perp)^2 = \frac{j}{2} + \frac{2j-1}{2j} \int d^2\Omega P(\Omega) \langle j, \Omega | j_\perp | j, \Omega \rangle^2, \qquad (2.19)$$

so that the SU(2) squeezing criterion (2.10) for $j > 1/2$ is equivalent to

$$\int d^2\Omega P(\Omega) \langle j, \Omega | j_\perp | j, \Omega \rangle^2 < 0. \qquad (2.20)$$

Since $\langle j, \Omega | j_\perp | j, \Omega \rangle^2$ is a positive function we get that SU(2) squeezing implies that $P(\Omega)$ cannot be a classical probability distribution.

D. Classical bounds

We derive classical upper bounds for the statistics of the measurement of arbitrary spin observables. This will be further particularized to the statistics of angular-momentum components in Sec. III.

1. *Bounds on the statistics of classical measurements*

For classical measurements the SU(2) P representative of the POVM element Δ_k is an ordinary nonnegative function $P_k(\Omega) \geq 0$ so that for every Ω

$$P_k(\Omega) Q(\Omega) \leq P_k(\Omega) Q_{\max}, \qquad (2.21)$$

where $Q_{\max}$ is the maximum of the Q function of the measured state (note that $Q(\Omega)$ is always a positive well behaved function). Applying this to the first equality in Eq. (2.9) we get the following upper bound for the statistics p_k of classical measurements [9]

$$p_k \leq \frac{4\pi}{2j+1} Q_{\max} \mathrm{tr} \Delta_k = \langle j, \Omega | \rho | j, \Omega \rangle_{\max} \mathrm{tr} \Delta_k = \tilde{\mathcal{P}}_k, \qquad (2.22)$$

where for finite-dimensional systems $\mathrm{tr}\Delta_k$ is always finite. Equation (2.22) can be violated if $P_k(\Omega)$ is more singular than a delta function or takes negative values. In both cases Eqs. (2.21) and (2.22) fail to be true. Therefore, the violation of condition (2.22) is a signature of nonclassical measurement.

2. Bounds on the statistics of classical states

Next we derive an upper bound for the probability of any outcome k that is to be satisfied by all classical states being measured, so that its violation becomes a sufficient (but not necessary) criterion of nonclassical behavior concerning the observed state. For classical states $P(\Omega)$ is an ordinary nonnegative function so that

$$P(\Omega)Q_k(\Omega) \leq P(\Omega)Q_{k,\max}, \tag{2.23}$$

where $Q_{k,\max}$ is the maximum of the Q function $Q_k(\Omega)$ of the POVM element Δ_k. Applying this to the last equality in Eq. (2.9) we get the following upper bound for the probability p_k of the outcome k

$$p_k \leq \frac{4\pi}{2j+1}Q_{k,\max} = \langle j,\Omega|\Delta_k|j,\Omega\rangle_{\max} = \mathcal{P}_k. \tag{2.24}$$

that holds for every $P(\Omega)$ compatible with classical physics. If this condition is violated for any k the state is not classical.

III. NONCLASSICALITY IN THE MEASUREMENT OF ANGULAR-MOMENTUM COMPONENTS

Next we apply the above approach to the particular case of the measurement of angular-momentum components. By SU(2) symmetry we can choose any component without loss of generality, say j_3. In such a case $\Delta_m = |j,m\rangle\langle j,m|$ with $\mathrm{tr}\Delta_m = 1$ so that the upper bound for classical measurements is

$$p_{j,m} \leq \langle j,\Omega|\rho|j,\Omega\rangle_{\max} = \tilde{\mathcal{P}}_{j,\rho}, \tag{3.1}$$

where ρ is the state being measured, and the upper bound for classical states is

$$p_{j,m} \leq |\langle j,m|j,\Omega\rangle|^2_{\max} = \mathcal{P}_{j,m}. \tag{3.2}$$

Note that both classical bounds are formally identical. From now on we consider $m \neq \pm j$, since otherwise $|j,m=\pm j\rangle$ are SU(2) coherent states and the bound for classical states is trivial $\mathcal{P}_{j,m} = 1$. On the other hand, since the states $|j,m=\pm j\rangle$ are angular-momentum classical they define a classical measurement and the bound (3.1) can never be surpassed.

The maximum of

$$|\langle j,m|j,\Omega\rangle|^2 = \begin{pmatrix} 2j \\ m+j \end{pmatrix} \sin^{2(j-m)}\left(\frac{\theta}{2}\right)\cos^{2(j+m)}\left(\frac{\theta}{2}\right) \tag{3.3}$$

when θ is varied is obtained for

$$\tan^2\frac{\theta}{2} = \frac{j-m}{j+m}, \tag{3.4}$$

so that the upperbound for the statistics of classical states is

$$\mathcal{P}_{j,m} = \begin{pmatrix} 2j \\ j+m \end{pmatrix}\left(\frac{j-m}{2j}\right)^{j-m}\left(\frac{j+m}{2j}\right)^{j+m}. \tag{3.5}$$

A. The measurement of angular-momentum components is nonclassical

In Eq. (3.1) let us consider that the measured state is equal to the measuring state, $\rho = \Delta_m = |j,m \neq \pm j\rangle\langle j,m\neq\pm j|$, so that the probability is unity $p_{j,m}=1$. On the other hand, the maximization in Eq. (3.1) is exactly the same we have just carried out so that the upperbound for the statistics of classical measurements is

$$\tilde{\mathcal{P}}_{j,m} = \begin{pmatrix} 2j \\ j+m \end{pmatrix}\left(\frac{j-m}{2j}\right)^{j-m}\left(\frac{j+m}{2j}\right)^{j+m}. \tag{3.6}$$

The minimum upper bound is obtained for $m=0$ for integer j and $m=\pm 1/2$ for half integer j. These outcomes are the best candidates to observe nonclassicality. More specifically, for integer j and $m=0$ we get

$$\tilde{\mathcal{P}}_{j,m=0} = \frac{(2j)!}{j!^2 2^{2j}} \simeq \frac{1}{\sqrt{\pi j}}, \tag{3.7}$$

where the approximation holds for $j \gg 1$. In this case the upper bound $\tilde{\mathcal{P}}_{j,m=0}$ is clearly below 1, so that Eqs. (3.1) and (3.6) are infringed and the measurement is not classical.

As a further example let us consider that the measured state is a classical state such as the equatorial phase-averaged SU(2) coherent state

$$\begin{aligned}\rho &= \frac{1}{2\pi}\int_{2\pi} d\phi |j,\theta=\pi/2,\phi\rangle\langle j,\theta=\pi/2,\phi| \\ &= \frac{1}{2^{2j}}\sum_{m=-j}^{j}\begin{pmatrix} 2j \\ j+m \end{pmatrix}|j,m\rangle\langle j,m|, \end{aligned}\tag{3.8}$$

where $|j,\theta=\pi/2,\phi\rangle$ are the corresponding equatorial SU(2) coherent states. In this case $\langle j,\Omega|\rho|j,\Omega\rangle_{\max}$ is obtained for $\theta=\pi/2$ for any ϕ, so that the classical upper bound (3.6) becomes

$$\tilde{\mathcal{P}}_{j,\rho} = \langle j,\Omega|\rho|j,\Omega\rangle_{\max} = \frac{1}{2^{4j}}\sum_{m=-j}^{j}\begin{pmatrix} 2j \\ j+m \end{pmatrix}^2, \tag{3.9}$$

while the statistics is

$$p_{j,m} = \langle j,m|\rho|j,m\rangle = \frac{1}{2^{2j}}\begin{pmatrix} 2j \\ j+m \end{pmatrix}. \tag{3.10}$$

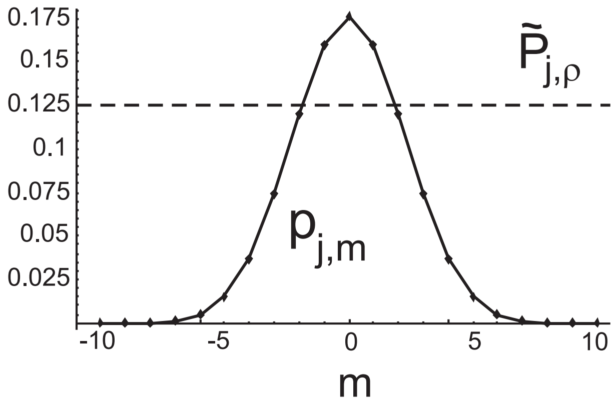

FIG. 1: Plot of $p_{j,m}$ (diamonds joined by a solid line) and $\tilde{\mathcal{P}}_{j,\rho}$ (dashed line) for a phase averaged SU(2) coherent state with $j = 10$ showing that the classical bound is clearly infringed by the statistics of the outcomes $m = 0, \pm 1$.

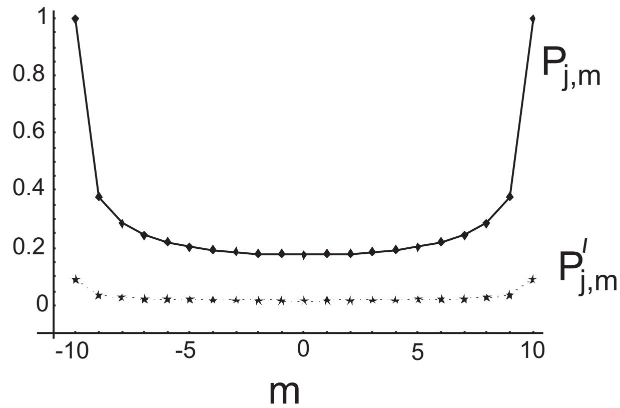

FIG. 2: Plot of $\mathcal{P}_{j,m}$ (diamonds joined by a solid line) and $\mathcal{P}'_{j,m}$ (stars joined by a dotted line) for $j = 10$ as functions of m to illustrate their difference.

In Fig. 1 we have represented $p_{j,m}$ (diamonds joined by a solid line) along with $\tilde{\mathcal{P}}_{j,\rho}$ (dashed line) for $j = 10$, showing that the classical bound is infringed by the probabilities of the outcomes $m = 0, \pm 1$. For example, for $m = 0$ we have $p_{j=10,m=0} = 0.18$ while $\tilde{\mathcal{P}}_{j=10,\rho} = 0.12$, so that the classical upper bound is infringed by a 50 % . As a further example, for $j = 1$ we get $p_{j=1,m=0} = 0.5$, while $\tilde{\mathcal{P}}_{j=1,\rho} = 0.37$.

B. SU(2) bounds are different from bosonic bounds

Let us focus on the bounds for classical states via measurement of an angular-momentum component in Eq. (3.5). These SU(2) bounds $\mathcal{P}_{j,m}$ are different from the bounds $\mathcal{P}'_{j,m}$ for the same statistics derived from quadrature P and Q functions associated to the bosonic realization (2.3). This was obtained in Eq. (5.5) of Ref. [9] as

$$\mathcal{P}'_{j,m} = \frac{(j+m)^{j+m}}{(j+m)!} \frac{(j-m)^{j-m}}{(j-m)!} \exp(-2j). \tag{3.11}$$

To illustrate this difference in Fig. 2 we have represented $\mathcal{P}_{j,m}$ (diamonds joined by a solid line) and $\mathcal{P}'_{j,m}$ (stars joined by a dashed line) for $j = 10$ as functions of m. It is shown that the SU(2) bounds are clearly above the quadrature bounds.

The relative difference increases when j increases. This can be easily seen in the case of integer j and $m = 0$ for example, so that

$$\mathcal{P}_{j,0} = \frac{(2j)!}{2^{2j} j!^2}, \quad \mathcal{P}'_{j,0} = \frac{j^{2j} \exp(-2j)}{j!^2}, \tag{3.12}$$

so that for $j \gg 1$

$$\mathcal{P}_{j,0} \simeq \frac{1}{\sqrt{\pi j}} \gg \mathcal{P}'_{j,0} \simeq \frac{1}{2\pi j}. \tag{3.13}$$

These bounds are different because they focus on information about different variables. As a simple illustrative example let us consider the case where both the measuring and measured state are the same SU(2) coherent state $\rho = \Delta_j = |j, m = j\rangle\langle j, m = j|$. In this case $p_{j,j} = \mathcal{P}_{j,j} = 1$ while

$$\mathcal{P}'_{j,j} = \frac{(2j)^{2j}}{(2j)!} \exp(-2j) \simeq \frac{1}{2\sqrt{\pi j}}, \tag{3.14}$$

where the approximation holds for $j \gg 1$. Therefore the quadrature bound for classical states $\mathcal{P}'_{j,j}$ is infringed, while the SU(2) bound $\mathcal{P}_{j,j}$ is not. The state $|j, m = j\rangle$ is clearly not classical concerning photon number statistics (strongly sub-Poissonian), but this is classical concerning SU(2) properties, as revealed for example in two-beam interferometry where these states just reach the standard quantum limit [21].

C. Independence of SU(2) squeezing and oscillatory statistics

Let us present an example of violation of the upper bounds for classical states without any other typical nonclassical behavior such as SU(2) squeezing of the orthogonal components $j_\perp$, nor oscillatory statistics of the measured observable j_3. To this end let us consider the measured state for integer $j > 2$

$$|\psi\rangle = \alpha|j, j\rangle + \beta|j, 0\rangle, \tag{3.15}$$

with $|\alpha|^2 + |\beta|^2 = 1$, while the measurement is $\Delta_0 = |j, 0\rangle\langle j, 0|$. The violation of the upper bound for classical states (3.5) holds when

$$p_{j,0} = |\beta|^2 > \frac{1}{2^{2j}} \binom{2j}{j}. \tag{3.16}$$

Let us apply to this state the most general SU(2) squeezing criterion in Eq. (2.10). For all $\alpha \neq 0$ the

most general $j_\perp$ is of the form

$$j_\perp = \cos\theta j_1 + \sin\theta j_2. \tag{3.17}$$

To compute $\langle\psi|j_\perp^2|\psi\rangle$ let us resort to the bosonic realization (2.3) so that

$$j_\perp = \frac{1}{2}\left(a_2^\dagger a_1 e^{i\theta} + a_1^\dagger a_2 e^{-i\theta}\right), \tag{3.18}$$

and, taking in this case $n = j$ since j is integer,

$$|\psi\rangle = \alpha|2n\rangle|0\rangle + \beta|n\rangle|n\rangle. \tag{3.19}$$

This allows us to conclude easily that for all θ

$$(\Delta j_\perp)^2 = \frac{1}{2}\left(|\beta|^2 j^2 + j\right) \geq \frac{j}{2}, \tag{3.20}$$

so that the weakest squeezing criterion (2.10) is never satisfied. Besides, there is no oscillatory statistics of the measured observable j_3 since there are just two outcomes $m = 0, j$.

D. SU(2) Schrödinger cat states

This is the coherent superposition of antipodal SU(2) coherent states, also known as NOON states [22]. In the $|j,m\rangle$ and photon number $|n_1\rangle|n_2\rangle$ bases they can be expressed as

$$|\psi\rangle = \frac{1}{\sqrt{2}}\left(|j,j\rangle + |j,-j\rangle\right) = \frac{1}{\sqrt{2}}\left(|n\rangle|0\rangle + |0\rangle|n\rangle\right), \tag{3.21}$$

with $j = n/2$. In this case the nonclassical behavior is revealed by the statistics of j_1

$$p_{j,m} = \frac{2}{2^{2j}}\binom{2j}{j+m} \tag{3.22}$$

for even $j+m$ and $p_{j,m} = 0$ otherwise. In Fig. 3 we have represented $p_{j,m}$ (diamonds joined by solid line) and the SU(2) bound for classical states $\mathcal{P}_{j,m}$ (stars joined by a dotted line) for $j = 10$. The plot shows that for $m = 0, \pm 2$ there is a clear violation of the classical state upper bounds. In particular, for $m = 0$ we get $p_{j=10,m=0} = 0.35$, while $\mathcal{P}_{j=10,m=0} = 0.18$, so that the classical upper bound is infringed by a 100 %.

The nonclassical behavior can be ascribed in this case to the oscillatory statistics of the measured observable j_1 as a result of the interference of probability amplitudes in the coherent superposition in Eq. (3.21). The interference minima $p_{j,m} = 0$ are compensated by the maxima, where $p_{j,m}$ takes twice the value for the corresponding SU(2) coherent state. Thus, the vanishing of $p_{j,m}$ for some m forces the other $p_{j,m}$ to raise above the classical limit.

Concerning SU(2) squeezing we have that $\langle \boldsymbol{j}\rangle = \boldsymbol{0}$, so that there is no parallel nor orthogonal components and the above squeezing criteria fail to be defined. Anyway,

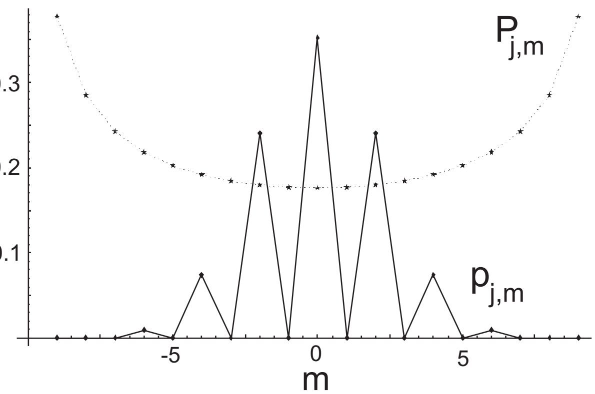

FIG. 3: Plot of the j_1 statistics $p_{j,m}$ (diamonds joined by solid line) for the Schrödinger cat state (3.21) and the SU(2) bound for classical states $\mathcal{P}_{j,m}$ (stars joined by a dotted line) in Eq. (3.5) for $j = 10$ showing that for $m = 0, \pm 2$ there is a clear violation of the classical upper bound.

the weakest squeezing criterion (2.10) is not satisfied for any component since

$$(\Delta j_3)^2 = j^2, \quad (\Delta j_{1,2})^2 = j/2, \tag{3.23}$$

as it can be easily computed using the bosonic realization (2.3). Nevertheless, these states provide better interferometric resolution than coherent states of the same mean number of photons [21, 22].

E. SU(2) intelligent squeezed states

Let us show that the intelligent states (2.14) satisfying squeezing criterion (2.13) violate classical state bounds. In the basis of eigenstates of $j_{\perp,1}$ the solution of Eq. (2.14) is [13]

$$|j,\eta\rangle = \mathcal{N}\sum_{m=-j}^{j}\binom{2j}{j+m}^{-1/2}\left[\frac{4(1-\eta^2)}{\eta^2}\right]^{(j+m)/2} P_{j+m}^{(-m,-m)}\left(\frac{1}{\sqrt{1-\eta^2}}\right)|j,m\rangle, \tag{3.24}$$

where $\mathcal{N}$ is a normalization constant and $P_\ell^{(m,n)}(x)$ are the Jacobi polynomials.

In Fig. 4 we have represented the statistics of $j_{\perp,1}$ (diamonds joined by solid line) for $j = 10$, $\eta = 0.5$

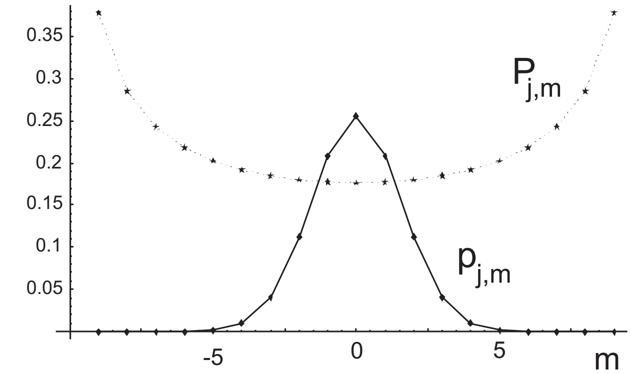

FIG. 4: Plot of the statistics of $j_{\perp,1}$ (diamonds joined by solid line) for the state (3.24) for $j = 10$, $\eta = 0.5$ along with the SU(2) classical state upper bound (3.5) (stars joined by dotted line) showing the violation of the classical bounds for small $|m|$.

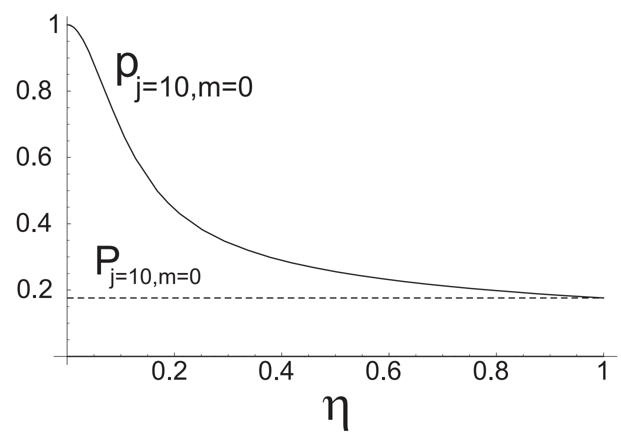

FIG. 5: Plot of the probability $p_{j,m=0}$ (solid line) of the eigenvalue $m = 0$ of $j_{\perp,1}$ for the state (3.24) with $j = 10$ as a function of η along with the SU(2) classical upper bound (3.6) $\mathcal{P}_{j,m=0}$ (dashed line) showing nonclassical behavior for all $\eta < 1$.

along with the SU(2) upper bound for classical states (3.5) (stars joined by dotted line) showing nonclassical behavior for $m = 0, \pm 1$. In particular for $m = 0$ we have $p_{j=10,m=0} = 0.26$ while the classical state bound is $\mathcal{P}_{j=10,m=0} = 0.18$, this is a 44 % violation of the classical bound.

In Fig. 5 we have plotted the probability $p_{j=10,m=0}$ (solid line) for the state (3.24) as a function of η along with the SU(2) classical state upper bound (3.6) $\mathcal{P}_{j=10,m=0}$ (dashed line) showing nonclassical behavior for all $\eta < 1$. The state tends to be classical as $\eta \to 1$ since in such a case it approaches an SU(2) coherent state.

F. Bright limit

Next we derive suitable formulas for the limit of a large number of photons $j \gg 1$. Besides we focus on the most favorable cases to violate the classical state upper bounds, this is $|m| \ll j$. By using the Stirling approximation we get the following bright limit for the classical bound $\mathcal{P}_{j,m}$ in Eq. (3.5)

$$\mathcal{P}_{j,m} \simeq \sqrt{\frac{j}{\pi(j^2 - m^2)}} \simeq \frac{1}{\sqrt{\pi j}}. \tag{3.25}$$

For $j \gg 1$ the discrete outcomes m are better described by a continuous variable x, so that j_1 for instance behaves like a single-mode quadrature operator X [12, 14, 23]

$$j_1 \simeq \sqrt{2j} X, \qquad m \simeq \sqrt{2j} x. \tag{3.26}$$

The probability distributions p_m and $p(x)$ are related in the form

$$p(x) \simeq \sqrt{2j} p_{j,m=\sqrt{2j}x}. \tag{3.27}$$

The corresponding classical upperbound for the statistics $p(x)$ derived from (3.25) and (3.27) are, respectively

$$p(x) \le \mathcal{P} = \sqrt{\frac{2}{\pi}}. \tag{3.28}$$

The bound $\mathcal{P}$ coincides with the bound for quadrature measurements derived from the quadrature P, Q functions [9]. This is to say that in this limit angular-momentum nonclassicality is equivalent to quadrature nonclassicality [12].

IV. CONCLUSIONS

We have provided feasible practical procedures to reveal the nonclassical behavior of angular-momentum states and measurements. Among other practical situations in quantum optics this includes two-beam interference and polarization.

A key point is that this approach refers exclusively to the nonclassical properties of angular momentum, being insensible to the nonclassical behavior of other variables such as total photon number. In this regard we have shown that the nonclassical test derived from SU(2) variables are more stringent than the one derived from the quadrature P, Q function for the same measurement.

The nonclassical tests proposed in this approach are exceedingly simple since definite conclusions are obtained without evaluation of moments, or any other more sophisticated data analysis. They are practical since they refer directly to the statistics of the measurement. Moreover, we have demonstrated that these nonclassical tests are independent of other typical quantum signatures such as SU(2) squeezing or oscillatory statistics.

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[1] L. Mandel and E. Wolf, *Optical Coherence and Quantum Optics* (Cambridge University Press, Cambridge, England, 1995); V. V. Dodonov, J. Opt. B: Quantum Semiclass. Opt. **4**, R1 (2002); U. M. Titulaer and R. J. Glauber, Phys. Rev. **140**, B676 (1965); L. Mandel, Phys. Scr. **T12**, 34 (1986); D.-G. Welsch, W. Vogel, and T. Opatrný, in *Progress in Optics*, edited by E. Wolf (Elsevier Science, Amsterdam, 1999), Vol. 39.

[2] M. Hillery, Phys. Rev. A **35**, 725 (1987).

[3] H. J. Kimble, M. Dagenais, and L. Mandel, Phys. Rev. Lett. **39**, 691 (1977); R. Short and L. Mandel, *ibid.* **51**, 384 (1983); R. E. Slusher, L. W. Hollberg, B. Yurke, J. C. Mertz, and J. F. Valley, *ibid.* **55**, 2409 (1985).

[4] D.N. Klyshko, Phys. Lett. A, **213**, 7 (1996); C.T. Lee, Phys. Rev. A **55**, 4449 (1997).

[5] G. S. Agarwal and K. Tara, Phys. Rev. A **46**, 485 (1992); A. Zavatta, V. Parigi, and M. Bellini1, *ibid.* **75**, 052106 (2007); T. Kiesel, W. Vogel, V. Parigi, A. Zavatta, and M. Bellini, *ibid.* **78**, 021804R (2008).

[6] W. Vogel, Phys. Rev. Lett. **84**, 1849 (2000); T. Richter and W. Vogel, Phys. Rev. Lett. **89**, 283601 (2002); E. Shchukin, T. Richter, and W. Vogel, Phys. Rev. A **71**, 011802(R) (2005); A. I. Lvovsky and J. H. Shapiro, *ibid.* **65**, 033830 (2002); E. V. Shchukin and W. Vogel, *ibid.* **72**, 043808 (2005); R. Alicki and N. Van Ryn, J. Phys. A **41**, 062001 (2008).

[7] R. W. Spekkens, Phys. Rev. Lett. **101**, 020401 (2008).

[8] A. Luis and L.L. Sánchez-Soto, Phys. Rev. Lett. **83**, 3573 (1999); J. Fiurášek, Phys. Rev. A **64**, 024102 (2001); H. B. Coldenstrodt-Ronge, J. S. Lundeen, K. L. Pregnell, A. Feito, B. J. Smith, W. Mauerer, Ch. Silberhorn, J. Eisert, M. B. Plenio, and I. A. Walmsley, J. Mod. Opt. **56**, 432 (2009); J. S. Lundeen, A. Feito, H. Coldenstrodt-Ronge, K. L. Pregnell, Ch. Silberhorn, T. C. Ralph, J. Eisert, M. B. Plenio, and I. A. Walmsley, Nature Physics **5**, 27 (2009).

[9] A. Rivas and A. Luis, Phys. Rev. A **79**, 042105 (2009).

[10] A. Luis and A. Rivas, Phys. Scr. **T143**, 014015 (2011).

[11] M. Kitagawa and M. Ueda, Phys. Rev. A **47**, 5138 (1993); D. J. Wineland, J. J. Bollinger, W. M. Itano, and D. J. Heinzen, *ibid.* **50**, 67 (1994); M. Hillery and L. Mlodinow, *ibid.* **48**, 1548 (1993); A. Sørensen, L.-M. Duan, J. I. Cirac, and P. Zoller, Nature (London) **409**, 63 (2001); S. Raghavan, H. Pu, P. Meystre, and N. P. Bigelow, Opt. Commun. **188**, 149 (2001); J. Hald, J. L. Sorensen, C. Schori, and E. S. Polzik, J. Mod. Opt. **47**, 2599 (2000).

[12] N. Korolkova and R. Loudon, Phys. Rev. A **71**, 032343 (2005); N. Korolkova, G. Leuchs, R. Loudon, T. C. Ralph, Ch. Silberhorn, *ibid.* **65**, 052306 (2002); J. Heersink, T. Gaber, S. Lorenz, O. Glöckl, N. Korolkova, and G. Leuchs, *ibid.* **68**, 013815 (2003).

[13] C. Brif and A. Mann, Phys. Rev. A **54**, 4505 (1996).

[14] A. Luis and N. Korolkova, Phys. Rev. A **74**, 043817 (2006).

[15] O. Giraud, P. Braun, and D. Braun, Phys. Rev. A **78**, 042112 (2008).

[16] A. Luis, Phys. Rev. A **71**, 053801 (2005); **73**, 063806 (2006).

[17] J. Schwinger, *Quantum Theory of Angular Momentum* (Academic Press, New York, 1965).

[18] M. O. Scully and M. S. Zubairy, *Quantum Optics* (Cambridge University Press, Cambridge, England, 1997).

[19] J. K. Korbicz, J. I. Cirac, and M. Lewenstein, Phys. Rev. Lett. **95**, 120502 (2005).

[20] F. T. Arecchi, E. Courtens, R. Gilmore, and H. Thomas, Phys. Rev. A **6**, 2211 (1972).

[21] C. M. Caves, Phys. Rev. D **23**, 1963 (1981); Z. Y. Ou, Phys. Rev. A **55**, 2598 (1997); V. Giovannetti, S. Lloyd, and L. Maccone, Science **306**, 1330 (2004).

[22] N. D. Mermin, Phys. Rev. Lett. **65**, 1838 (1990); J. J. Bollinger, W. M. Itano, D. J. Wineland, and D. J. Heinzen, Phys. Rev. A **54**, R4649 (1996); S. F. Huelga, C. Macchiavello, T. Pellizzari, A. K. Ekert, M. B. Plenio, and J. I. Cirac, Phys. Rev. Lett. **79**, 3865 (1997); A. Luis, Phys. Rev. A **64**, 054102 (2001); **65**, 034102 (2002); Ph. Walther, J.-W. Pan, M. Aspelmeyer, R. Ursin, S. Gasparoni, and A. Zeilinger, Nature **429**, 158 (2004); M. W. Mitchell, J. S. Lundeen, and A. M. Steinberg, *ibid.* **429**, 161 (2004).

[23] A. Rivas and A. Luis, Phys. Rev. A **77**, 022105 (2008); **78**, 043814 (2008).